

Studies on Negative Refractive Index without Absorption

R. M. Rajapakse,¹ E. Kuznetsova,^{1,2} and S. F. Yelin^{1,2}

¹*Department of Physics, University of Connecticut, Storrs, CT 06269*

²*ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138*

(Dated: October 31, 2012)

Which systems are ideal to obtain negative refraction with no absorption? Electromagnetically induced transparency (EIT) is a method to suppress absorption and make a material transparent to a field of a given frequency. Such a system has been discussed in [1]; however the main limitations for negative refraction introduced are the necessity of resonant electric and magnetic dipole transitions, and the necessity of very dense media. We suggest using frequency translators in a composite system that would provide negative refraction for a range of optical frequencies while attempting to overcome the limitations discussed above. In the process of using frequency translators, we also find composite systems that can be used for refractive index enhancement.

I. INTRODUCTION

Negative refraction has been a topic of interest since the 1960s [2] and has become an active area of research in modern photonics. The development of a perfect lens where the imaging resolution is not limited by wavelength is an application of much interest [2]. After the initial proposal for realization in meta-materials [3] and experimental demonstration [4] for RF radiation, much progress has been made towards negative refraction for shorter and shorter wavelengths [4–10]. The approaches include split-ring resonator meta-materials [4, 7, 8] and photonic crystals [9, 10]. Much development has been made in this field to date [4, 7, 8] but creating negative refraction with minimal absorption has been a challenge.

Although it was first suggested that negative refractive index required media with both negative permittivity and permeability ($\epsilon, \mu < 0$) in the frequency range of interest [2], when the optical regime is considered it is difficult to realize negative permeability without considerable loss. The transition magnetic dipole moments (μ_a) are smaller than the transition electric dipole moments d_a by a factor of the order of the fine structure constant α . As a result, the magnetic susceptibilities are much smaller than electric susceptibilities. This condition can be alleviated by using a material such as a chiral medium where the electric and magnetic properties are cross-coupled [11].

Enhancement of refractive index has also been a topic of interest [12], and experimental results have shown that such enhancement can be achieved with no absorption, using electromagnetically induced transparency (EIT) [12]. The use of EIT to suppress absorption has been suggested by refs [1, 13], and several systems which can be used to achieve electromagnetically induced negative refraction without absorption are discussed in this paper. We propose to extend the work done by [13].

The scheme used in [1, 13] is based on coupling the electric and magnetic components of the probe field to different transitions in the atom, one electric and one magnetic dipole transition. Thus, the main limitations of introducing atomic systems for negative refraction are the necessity of resonant electric and magnetic dipole

transitions, and the need of very dense media. This latter condition stems mostly from the fact that all elements in the scheme involving magnetic dipole coupling are by nature very weak. Many atomic, bound exciton or polar molecular schemes have few resonant levels for electric and magnetic dipole transitions, especially in the range of optical frequencies because magnetic quantum transitions require the same electronic configuration, i.e. the same n and l quantum numbers, which can for optical frequencies only be reached for systems with high relativistic effects, i.e., involving very heavy atoms.

In order to obtain sufficiently large magnetic dipole transition frequencies for example, to obtain a visible dipole transition, it is also necessary to use heavy atoms. Even so, as there are few resonant levels we attempt to implement a coherent frequency translator in order that electric and magnetic dipole coupling can occur between two levels with different optical frequencies while leaving the phase intact. We then revise the system which includes the frequency translator into a different level structure which decreases the absorption and has negative refraction over a larger range of detuning.

II. NEGATIVE REFRACTION VIA ELECTROMAGNETICALLY INDUCED CHIRALITY

We begin by revisiting some of the concepts of [1]. The refractive index of a medium can be expressed by the following equation:

$$n = \sqrt{\epsilon\mu - \frac{(\xi_{EH} + \xi_{HE})^2}{4}} - \frac{i}{2}(\xi_{EH} - \xi_{HE}) \quad (1)$$

where the values of ϵ , μ , ξ_{HE} and ξ_{EH} can be found from Eq. (2) [14].

A simple three level system (see Fig. 1(a)) can be used to introduce electromagnetically induced chiral negative refraction. Here the electric and magnetic components of the probe field couple state $|1\rangle$ to $|2\rangle$ by an electric dipole transition and state $|3\rangle$ to state $|2\rangle$ by a magnetic dipole transition. a strong coherent field couples $|1\rangle$ and

$|3\rangle$ with Rabi frequency Ω_c . Level $|3\rangle$ can be considered metastable with a decay rate $\gamma_3 \sim (1/137^2) \gamma_1$.

In realistic media this setup can be modified using three criteria (i) Ω_c must be an ac field so its phase can be adjusted to induce negative refraction. (ii) There must be high contrast EIT for the probe field in order to suppress absorption and (iii) the energy level structure must be appropriate for media of interest as the parities should be conducive to introducing the appropriate electric and magnetic dipole coupling. Fig. 1(b) is an example that meets the above criteria. This scheme employs strong coherent Raman coupling by two coherent fields with complex Rabi fields Ω_1 and Ω_2 and carrier frequencies of ω_1 and ω_2 , creating a superposed dark state of the two levels $|1\rangle$ and $|4\rangle$. The dark state acts as the ground state and the modified scheme is effectively the same as with Fig. 1(a) (which is taken from [1]) but with more degrees of control.

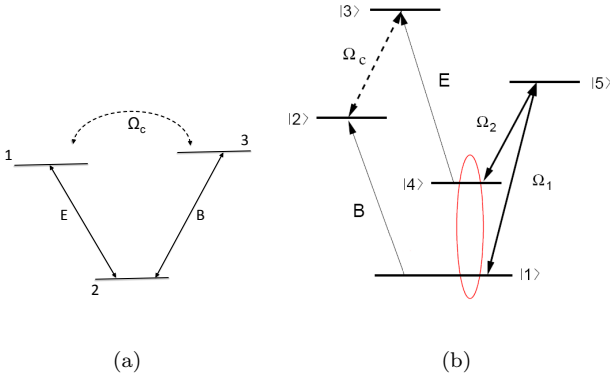


FIG. 1: a) Level scheme for obtaining electromagnetically induced chirality. b) Modified level scheme for obtaining electromagnetically induced chirality [1]. The dark state created by the two-photon resonant Raman coupling of levels $|1\rangle$ and $|4\rangle$ takes over the role of level $|1\rangle$ in (a). Both level schemes have E and B as the electric and magnetic components of the probe field, which can experience negative refraction.

III. LIMITATIONS OF CURRENT APPROACH

For the scheme described in Fig. 1(b) we use a medium which has the cross couplings given in Eq. (2) [14] to obtain negative refraction, with

$$\begin{aligned} \mathbf{P} &= \epsilon_0 \chi_e \mathbf{E} + \frac{\xi_{EH}}{c} \mathbf{B} \\ \mathbf{M} &= \frac{\xi_{HE}}{\mu_0 c} \mathbf{E} + \chi_m \mathbf{B} \end{aligned} \quad (2)$$

The above equations require that there should be a coupling between the electric and magnetic fields in the medium, which have to be components of the same input field.

The requirement of a single input field imposes some limitations: the levels being coupled by the electric field and the levels coupled by the magnetic field should then have the same energy difference and opposing parities. Level selection in such a situation is complicated; magnetic dipole transitions leave electronic configurations intact and it is difficult to find a material in which this condition is satisfied. Heavy atoms have relativistic effects which satisfy the conditions of changing the electronic configuration. Rare-earth materials such as Dysprosium vapor, Scandium and Neodymium, therefore, have a level structure that is conducive to obtaining negative refraction [20].

The two problems of a difficult level selection and the requirement of high densities, which are of the order of 10^{17}cm^{-3} for typical parameters [1] have to be solved to extend the types of materials which can be used to obtain negative refraction. We consider composite systems which involve frequency conversion to overcome the requirement of having the electric and magnetic field have the same frequency.

IV. COMPOSITE SYSTEMS FOR WIDER PARAMETER REGIONS

To overcome the difficulty of finding a system where an allowed electric dipole transition is the exact same frequency as an allowed magnetic dipole transition, we attempt to expand our scheme in a way that allows dipole coupling of electric and magnetic components of different fields, but yet retains characteristics, particularly phase information of the original single field entering the media. Conceptually, the index of refraction indicates the response of a field to the initial atomic response, or the changes in phase velocity of the field. As the field has an effect on the phases of both polarization and magnetization, the interplay of the electric and magnetic component of the field has to be taken into account. In order to achieve the result described above it is necessary to use a single initial field, divide it so that the phase is maintained, and later recombine the fields coherently and observe the response.

a. Proof of principle

We require the one-to-one conversion of one field to another down to the single photon level model, while keeping the phase and coherence unchanged. The calculation below illustrates that adding frequency conversion does not fundamentally change the dynamics of a single photon from the five level process described in [1]. H_{NI} is the Hamiltonian for Fig. 1(b), which describes the dynamics of a single photon in a five level medium.

$g_1^* a^\dagger \sigma_{34}$ and $g_2^* a^\dagger \sigma_{21}$ describe the coupling of the probe field to the two probe transitions. $g_1^* a^\dagger \sigma_{34}$ is an electric dipole transition while $g_2^* a^\dagger \sigma_{21}$ is a magnetic dipole

transition. We obtain the single-photon Hamiltonian for Fig. 1(b), which gives us:

$$H_{NI} = \sum_{i=1}^5 \Delta_i \sigma_{ii} + \delta a^\dagger a + g_1^* a^\dagger \sigma_{34} + g_2^* a \sigma_{21} + \Omega \sigma_{23} + \Omega_1 \sigma_{45} + \Omega_2 \sigma_{15} + \text{h.c.} \quad (3)$$

From the above, we obtain the following dynamics:

$$\partial a = \imath \delta a + \imath g_1^* \sigma_{34} + \imath g_2^* \sigma_{21}. \quad (4)$$

The quantity δ is related to the dispersion mismatch between pumps and sidebands [15].

One idea to create a division of a single field while maintaining the phase is to add a four wave mixing process in order to ensure that there is coupling between two fields that enter the four wave mixing medium and the fields of different frequency that exit it [16, 17]. We use four wave mixing in a composite system, and discuss how it affects the behavior of a single photon.

For the composite system with frequency conversion, our Hamiltonian changes to

$$\begin{aligned} H &= H_{mix} + H_{NI} \\ H_{mix} &= \delta_1 a_1^\dagger a_1 + \delta_2 a_2^\dagger a_2 + \kappa a_1^\dagger a_2 + \kappa^* a_2^\dagger a_1 \\ H_{NI} &= \sum_{i=1}^5 \Delta_i \sigma_{ii} + g_1^* a_1^\dagger \sigma_{34} + g_2^* a_2^\dagger \sigma_{21} + \Omega \sigma_{23} + \Omega_1 \sigma_{45} + \Omega_2 \sigma_{15} + \text{h.c.} \end{aligned} \quad (5)$$

By eliminating a_2 adiabatically, this gives us

$$\partial a_1 = \imath \left(\delta_1 - \frac{|\kappa|^2}{\delta_2} \right) a_1 + \imath g_1^* \sigma_{34} - \imath \frac{\kappa}{\delta_2} g_2^* \sigma_{21} \quad (6)$$

where κ is related to the nonlinearity [15]. From the similarity of Eq. (4) and Eq. (6) above, we can see that the addition of a frequency conversion Hamiltonian still gives us an outcome which is analogous to that of a single photon with no frequency conversion. We use four wave mixing for the physical implementation of this frequency conversion.

b. Four wave mixing

Four wave mixing involves four electromagnetic fields interacting in a medium. To keep the relationship between the electric and magnetic components of the field intact in a negative refractive media, we use an initial weak single field to create two fields of different frequencies that coherently couple both an electric dipole transition and a magnetic dipole transition in the medium we use. This will ensure that two different fields are created that will carry the same phase information, and these fields will create separate electric and magnetic dipole transitions in the negative refractive part of the system corresponding to Fig. 1(b), with a Hamiltonian of similar form as Eq. (3).

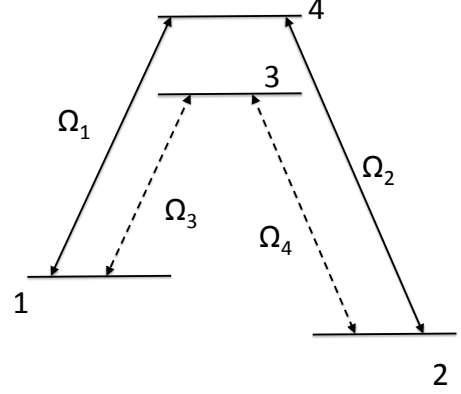


FIG. 2: A four wave mixing setup to obtain an additional field. One exiting field creates an electric dipole transition, and the other a magnetic dipole transition in Fig. 1(b)

Our suggested four level mixing medium consists of a double Λ system. Two control fields in the four level system, Ω_1 and Ω_2 are the driving fields which keep the population in the mixed ground state, and the third anti-Stokes field is a field Ω_3 which couples levels $|1\rangle$ and $|4\rangle$ in Fig. 2.

Mixing of these three fields creates a Stokes fourth field, Ω_4 which will then couple two magnetic dipole allowed levels, with a frequency $\omega_4 = \omega_1 + \omega_2 - \omega_3$. We specify that the two weak fields couple to a single upper level in order to minimize gain.

For the output, we consider two input fields (the original and the generated) that pass through another four wave mixing setup.

All the fields must maintain a constant phase relative to the others:

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \quad (7)$$

where any change in relative phase corresponds to a change in the frequency of the outgoing fields and is therefore not conducive to absorption-free four wave mixing. Fig. 3(a) shows the setup we suggest.

In materials of positive refractive index the Poynting vector is in the same direction as \vec{k} but in negative index materials the Poynting vector moves in the opposite direction to \vec{k} [18]. Additional error occurring in four wave mixing because of phase noise is discussed in detail in [19]. Atomic noise is contributed by the relaxation rates of the ground state coherence, and an additional noise contribution occurs from the absorption of the driving field. Minimizing absorption within the four-wave mixing setup can be achieved by adding a detuning of one of

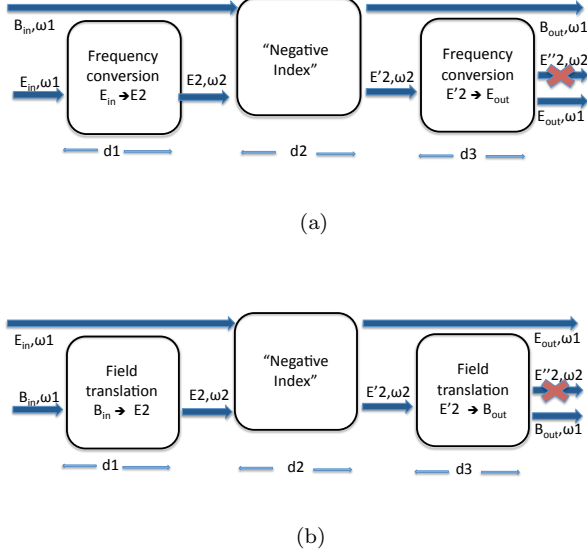


FIG. 3: Two composite setups. Fig.3(a) translates the frequency of the electric field, while Fig.3(b) has translation between electric and magnetic fields.

the driving fields from resonance and choosing the driving fields within certain parameters as calculated in [19].

V. SIMULATION: EFFECTIVE REFRACTIVE INDEX

The "effective refractive index" of a mixed system depends on the effective electric and magnetic susceptibilities of the complete system. In order to derive the effective refractive index of the material, we have to obtain the effective permittivity, the effective permeability and the effective cross-couplings of the combined system. The effective susceptibilities given below are calculated assuming a composite system as in Fig.3(a) and Fig.3(b), where all the transitions take place in a single combined system of thickness d , where $d = d_1 + d_2 + d_3$. We calculate the propagation matrices for the fields E_{in} and B_{in} in Fig.3(a) and find the equivalent matrix that transforms the input field to the output field. Detailed derivations are given in Appendix A.

The effective susceptibilities of the composite system

are given by

$$\begin{aligned}\chi_{ee-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)} (C_2 C_5 d_1 d_3 \chi_{21} \chi_{12} (1 + d_2 F_4 \chi_{ee}) - 1) \\ \chi_{eb-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)} d_1 d_2 F_4 C_5 \chi_{eb} \chi_{12} \\ \chi_{bb-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)} (d_2 F_3 \chi_{bb}) \\ \chi_{be-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)} d_1 d_2 C_2 F_3 \chi_{be} \chi_{21}\end{aligned}\quad (8)$$

Here d_1 , d_2 and d_3 are the thicknesses of the materials in the composite system, $C_2 = F_4 = \frac{i\omega_2}{c}$ and $C_5 = F_3 = \frac{i\omega_1}{c}$.

We then use the effective susceptibilities in Eq. (1) to obtain the refractive index.

a. Results

The results of this calculation are given in Fig. 4. We can obtain negative refraction using this composite setup. Negative refraction is achieved at lower densities than if we used a single five level system as in [1], which is shown in Fig. 6.

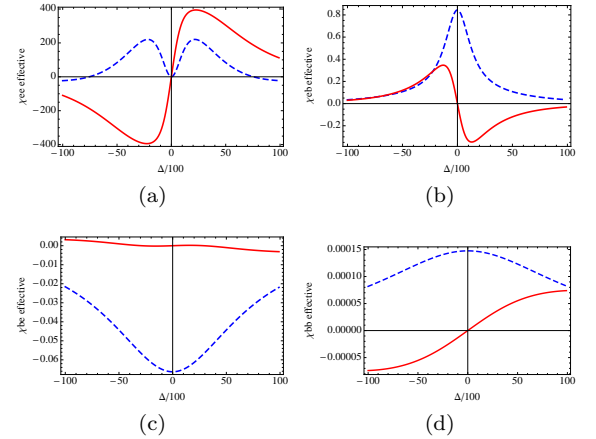


FIG. 4: (a)-(d) Real(red) and imaginary(blue-dashed) parts of the effective susceptibilities for the composite system with frequency translation of the electric field. Density of "negative refractive" system is $9 \times 10^{15} \text{ cm}^{-3}$, $d_1 = d_3 = 8 \times 10^{-2} \text{ cm}$, $d_2 = 8 \text{ cm}$.

b. Calculating effective refraction with translation between electric and magnetic fields

We consider the system given in Fig. 3(b), with a field of frequency ω_1 which is once again changed to ω_2 . However in this system ω_2 couples an electric dipole transition instead of a magnetic dipole transition, and the results

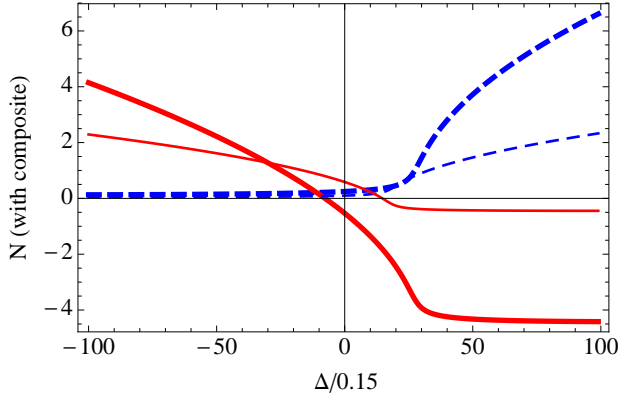


FIG. 5: Refractive index of the system with frequency translation with density of "negative refractive" system $9 \times 10^{15} \text{cm}^{-3}$ (thick lines) and $1 \times 10^{15} \text{cm}^{-3}$ (thin lines). Red - real part of refractive index and blue, dashed - imaginary part of refractive index

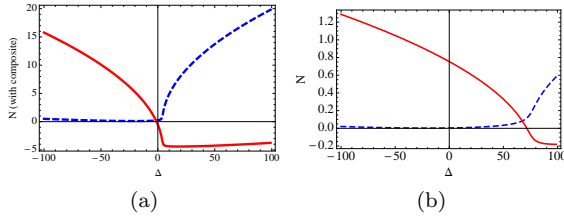


FIG. 6: Real (red) and imaginary (blue-dashed) refractive indices (a) for a composite system and (b) for a single five level system. The density of the "negative refractive" system is $9 \times 10^{15} \text{cm}^{-3}$ and the density of the four wave mixing system is $8 \times 10^{12} \text{cm}^{-3}$. $d_1 = d_3 = 8 \times 10^{-2} \text{cm}$, $d_2 = 8 \text{cm}$.

of our calculations are changed. The setup is given in Fig. 3(b).

A similar matrix calculation as in section VIIa gives us the following effective susceptibilities. There is cross coupling between fields E_1 and E_2 in the four level system which is a double Λ system. As EIT effects are required to suppress absorption the population in the ground states of the 4 level system have to be controlled. This can be accomplished via coupling the ground states to a fifth level.

The effective susceptibilities of this composite system are:

$$\begin{aligned}\chi_{ee-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)}(C_3 d_2 d_3 \chi_{22}) \\ \chi_{eb-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)} d_2 d_3 C_3 C_6 \chi_{eb} \chi_{21} \\ \chi_{bb-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)} (C_2 C_6 d_1 d_2 \chi_{be} \chi_{be} (1 + \\ &\quad C_4 d_2 \chi_{11}) \\ \chi_{be-eff} &= \frac{c}{i\omega_1(d_1 + d_2 + d_3)} d_1 d_2 C_2 C_4 \chi_{be} \chi_{12}\end{aligned}\quad (9)$$

Here d_1 , d_2 and d_3 are the thicknesses of the materials in the composite system, $C_2 = C_4 = C_6 = \frac{i\omega_2}{c}$ and $C_3 = \frac{i\omega_1}{c}$.

c. Results

The results of this calculation are given in Fig. 7. Using this system it is possible to obtain enhanced refractive indices with no absorption. For some detunings it is possible to obtain negative refraction, however in this composite system negative refraction comes at the cost of high absorption.

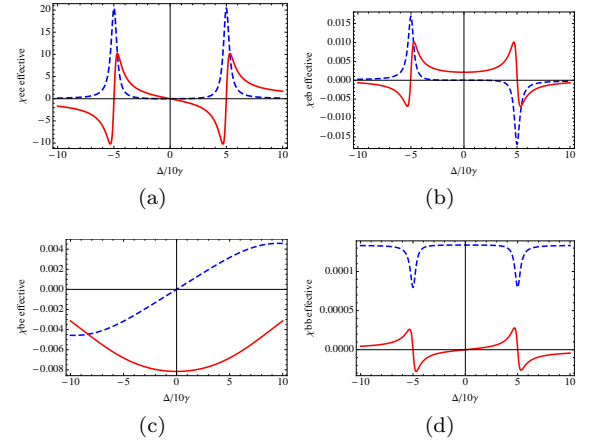


FIG. 7: (a)-(d) Real (red) and imaginary (blue-dashed) parts of the effective susceptibilities for the magnetic to electric field translational system with density of 4 level system $5 \times 10^{15} \text{cm}^{-3}$. The density of the translational system is $8 \times 10^{12} \text{cm}^{-3}$, and thicknesses are $d_1 = d_3 = 8 \times 10^{-2} \text{cm}$ and $d_2 = 8 \times 10^{-1} \text{cm}$.

VI. CONCLUSION

The types of materials that can be used for negative refraction has so far been restrained by the necessity of having level structures that have electric and magnetic dipole transitions of the same frequency. It was also necessary to obtain magnetic dipole transitions in the optical

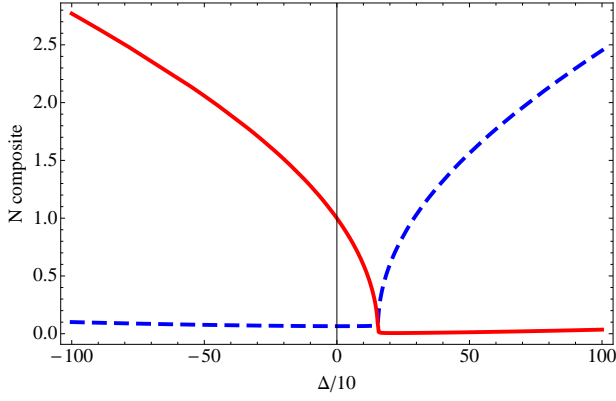


FIG. 8: Refractive index of the system with magnetic to electric field translation with 4 level scheme density $5 \times 10^{15} \text{cm}^{-3}$. Red - real part of refractive index and blue, dashed - imaginary part of refractive index

regime, which meant that heavy atoms with large level separation needed to be used as magnetic dipole coupling is much weaker compared to electric dipole coupling.

Our investigations show that we can find a combined scheme that generates additional fields in order to obtain the electric and magnetic dipole coupling that is necessary for electromagnetically induced chirality. We can thus obtain negative refractive index with minimal absorption for materials with a level structures fitting that of the combined system, including atoms, molecules and excitons. As the change of the level structure from a simple 5-level system to a composite system gives a great deal of flexibility to the medium we can select, electromagnetically induced chirality can be realized in a wide range of materials.

In conclusion, it can be understood that the scope of negative refractive index materials can be much increased by a combined level scheme in order to generate additional fields that create electric and magnetic dipole coupling that is necessary for electromagnetically induced chirality. The densities needed to obtain negative refraction are reasonable, and fit the densities of doped crystals and several rare earth material vapors.

VII. ACKNOWLEDGEMENTS

We thank M. Fleischhauer and J. Otterbach for useful discussions, and the NSF for funding.

Appendix A

a. Calculating effective refractive Index for a composite system with frequency translation

The incoming electromagnetic field E_{in} creates a field E_2 , and the exiting fields are given by E_1 and E_2 . For

this system:

$$\begin{aligned} P_1 &= \chi_{11}E_1 + \chi_{12}E_2 \\ P_2 &= \chi_{22}E_2 + \chi_{21}E_1 \\ M_1 &= 0 \\ M_2 &= 0 \end{aligned} \quad (10)$$

Here, $P_1 = \iota k_1 \frac{dE_1}{dz}$ and $P_2 = \iota k_2 \frac{dE_2}{dz}$. There is no magnetic dipole coupling.

If the thickness of the medium of five level system d_2 ,

$$\frac{dE}{dz} = \frac{\iota k_1}{2\epsilon_0} P, \quad (11)$$

If the length of the medium is assumed to be smaller than one wavelength, the following approximation holds:

$$\begin{aligned} \frac{\Delta E}{d} &= \iota \frac{k_1}{2\epsilon_0} P \\ \frac{E_{out} - E_{in}}{d} &= \iota \frac{k_1}{2\epsilon_0} P \end{aligned} \quad (12)$$

where P is the polarization density and d is the thickness. We calculate the magnetization density in a similar manner.

Using Eq.(12) we can calculate the propagation

matrix for the fields: $\begin{bmatrix} 1 & 0 \\ 0 & C_2 d_1 \chi_{21} \end{bmatrix}$

$$\text{so that } \begin{bmatrix} 1 & 0 \\ 0 & C_2 d_1 \chi_{21} \end{bmatrix} \begin{bmatrix} H_{in} \\ E_{in} \end{bmatrix} = \begin{bmatrix} H_1 \\ E_2 \end{bmatrix}$$

Here $C_2 = \omega_2/c$ where ω_2 is the frequency of the created field.

The fields H_1 and E_2 then enter the second part of the composite system where the polarization and magnetization can be written for the two different frequencies as follows:

$$\begin{aligned} P_A &= 0 \\ M_A &= \chi_{bbA}H_1 + \chi_{beA}E_2 \\ P_B &= \chi_{eeA}E_2 + \chi_{ebA}H_1 \\ M_B &= 0 \end{aligned}$$

where A and B are the two frequencies we consider. These equations can be rearranged and written in array form, again using the assumption of small thick-

ness, as: $\begin{bmatrix} \frac{H_1 - H_1}{d_2} \\ \frac{E_2 - E_2}{d_2} \end{bmatrix} = \begin{bmatrix} F_3 \chi_{beA} & F_3 \chi_{bbA} \\ F_4 \chi_{eeA} & F_4 \chi_{ebA} \end{bmatrix} \begin{bmatrix} H_1 \\ E_2 \end{bmatrix}$, which

gives us

$$\begin{bmatrix} H_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} F_3 d_2 \chi_{bbA} + 1 & + F_3 d_2 \chi_{beA} \\ F_4 d_2 \chi_{ebA} & F_4 d_2 \chi_{eeA} + 1 \end{bmatrix} \begin{bmatrix} H_1 \\ E_2 \end{bmatrix}$$

where $F_3 = \omega_1/c$ and $F_4 = \omega_2/c$.

The fields then enter a third system which is similar to the first, and the matrix for the third box can be written as $\begin{bmatrix} 1 & 0 \\ 0 & -C_2 d_3 \chi_{12} \end{bmatrix}$. By multiplying these matrices together we can find a matrix of the "effective" susceptibilities that transform the field E_{in} (B_{in}) to E_{out} (B_{out}).

Appendix B

b. Maxwell's equations for slowly varying field functions and change of the field phase

The electromagnetic field radiation when the medium has both polarization and magnetization can be written out as follows

$$\nabla \cdot \mathbf{D} = 0 \quad (13)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (14)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (15)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (16)$$

where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$, $\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$ and $\mathbf{J} = \sigma \mathbf{E}$.

We obtain the following results with the aid of Maxwell's equations:

$$-\nabla^2 \mathbf{E} = \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} + \mu_0 \frac{\partial(\nabla \times \mathbf{M})}{\partial t} \quad (17)$$

$$-\nabla^2 \mathbf{B} = \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \mu_0 \nabla^2 \mathbf{M} + \mu_0 \frac{\partial(\nabla \times \mathbf{P})}{\partial t} \quad (18)$$

where σ is the conductivity.

The x and y dependence of E, B P and M are neglected and we assume $\mathbf{E}(\mathbf{r}, t) = E_x(z, t)\hat{x} + E_y(z, t)\hat{y}$, $\mathbf{B}(\mathbf{r}, t) = B_x(z, t)\hat{x} + B_y(z, t)\hat{y}$, $\mathbf{P}(\mathbf{r}, t) = P_x(z, t)\hat{x} + P_y(z, t)\hat{y}$ and $\mathbf{M}(\mathbf{r}, t) = M_x(z, t)\hat{x} + M_y(z, t)\hat{y}$.

Considering equation Eq. (17), we then attempt to rewrite the equation separately in each of the vectorial directions. If we consider the x-direction, Eq. (17) becomes

$$-\frac{\partial^2 E_x}{\partial z^2} \hat{x} = \mu_0 \sigma \frac{\partial E_x}{\partial t} \hat{x} + \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \hat{x} + \mu_0 \frac{\partial^2 P_x}{\partial t^2} \hat{x} + \mu_0 \frac{\partial(\partial M_y / \partial z)}{\partial t} \hat{x} \quad (19)$$

Similarly, Eq. (18) becomes

$$-\frac{\partial^2 B_y}{\partial z^2} \hat{y} = \mu_0 \sigma \frac{\partial B_y}{\partial t} \hat{y} + \mu_0 \epsilon_0 \frac{\partial^2 B_y}{\partial t^2} \hat{y} - \mu_0 \frac{\partial^2 M_y}{\partial z^2} \hat{y} - \mu_0 \frac{\partial(\partial P_x / \partial z)}{\partial t} \hat{y} \quad (20)$$

in the y direction. At this point, we remove the directional subscripts and continue the derivations of Eq. (19) and Eq. (20).

We write the fields of frequency ν in the following form

$$E_x(z, t) \equiv \frac{1}{2} \tilde{E}^+(z, t) e^{-i[\nu t - kz]} + \frac{1}{2} \tilde{E}^-(z, t) e^{+i[\nu t - kz]} \\ B_y(z, t) \equiv \frac{1}{2} \tilde{B}^+(z, t) e^{-i[\nu t - kz]} + \frac{1}{2} \tilde{B}^-(z, t) e^{+i[\nu t - kz]} \quad (21)$$

We substitute this in Eq. (17) and Eq. (18) where $\phi(z, t)$ is also a slowly varying function of position and time. The response of the medium, neglecting higher harmonics can be given by the polarization

$$P_x(z, t) \equiv \frac{1}{2} \mathcal{P}^+(z, t) e^{-i[\nu t - kz]} + \frac{1}{2} \mathcal{P}^-(z, t) e^{+i[\nu t - kz]} \quad (22)$$

we can derive a similar equation for $M_y(z, t)$.

We make substitutions from Eq. (22), Eq. (21) and Eq. (24), and also use:

$$\mathcal{P}(z, t) v^2 \gg \frac{\partial^2 \mathcal{P}(z, t)}{\partial t^2}, \text{ and which gives us}$$

$$\frac{\partial^2 (\mathcal{P}^+ e^{-i[\nu t - kz]})}{\partial t^2} \approx -\nu^2 \mathcal{P}^+ e^{-i[\nu t - kz]} - 2i\nu \frac{\partial \mathcal{P}^+}{\partial t} e^{-i[\nu t - kz]} \quad (23)$$

We also make the assumption that $\tilde{E}^+ k^2 \gg \frac{\partial^2 \tilde{E}^+(z, t)}{\partial z^2}$ which gives us the result

$$\frac{\partial^2 \tilde{E}^+(z, t) e^{-i[\nu t - kz]}}{\partial z^2} \approx -k^2 \tilde{E}^+(z, t) e^{-i[\nu t - kz]} + 2ik \frac{\partial \tilde{E}^+(z, t)}{\partial z} e^{-i[\nu t - kz]}$$

Once again we make the assumption that $\tilde{E}^+(z, t) v^2 \gg \frac{\partial^2 \tilde{E}^+(z, t)}{\partial t^2}$ which gives us the result

$$\frac{\partial^2 \tilde{E}^+(z, t)}{\partial z^2} = -v^2 \tilde{E}^+(z, t) e^{-i[\nu t - kz]} + 2i\nu \frac{\partial \tilde{E}^+(z, t)}{\partial z} e^{-i[\nu t - kz]}$$

$\tilde{E}^+(z, t)$ ($\tilde{B}^+(z, t)$) is a slowly varying function and can be rewritten in terms of a real amplitude \mathcal{E} (\mathcal{B}) and a phase ϕ_E (ϕ_B)

$$\tilde{E}^+(z, t) = \mathcal{E}(z, t) e^{i[\phi_E(z, t)]} \\ \tilde{B}^+(z, t) = \mathcal{B}(z, t) e^{i[\phi_B(z, t)]} \quad (24)$$

The time and space derivatives of $\tilde{E}(z, t)$ can be derived thus:

$$\frac{\partial \tilde{E}^+(z, t)}{\partial z} = i \frac{\partial \phi}{\partial z} \mathcal{E}^+(z, t) e^{i\phi_E(z, t)} + \frac{\partial \mathcal{E}^+(z, t)}{\partial z} e^{i[\phi_E(z, t)]}$$

A similar derivation can be done for $\tilde{B}(z, t)$.

For ease of comprehension, the equations dealing with the electric field will be derived further; the equations of the magnetic fields follow the same derivation. Substituting Eq. (22) and Eq. (21) in Eq. (19) and adding the necessary subscripts to denote the directions, the resulting equations are

$$\begin{aligned}
k \frac{\partial \mathcal{E}_x}{\partial z} + \frac{\nu}{c^2} \frac{\partial \mathcal{E}_x}{\partial t} + k \mathcal{E}_x \frac{\partial \phi_{Ex}}{\partial z} + \frac{\nu}{c^2} \mathcal{E}_x \frac{\partial \phi_{Ex}}{\partial t} = & \kappa \mathcal{E} - \frac{1}{2\epsilon_0 c^2} \nu^2 \Im(\mathcal{P}_x) - \\
& \frac{k\nu}{2\epsilon_0 c^2} \Im(\mathcal{M}_y) + \mathcal{E} \left(k^2 - \frac{\nu^2}{c^2} \right) - \\
& \frac{1}{2\epsilon_0 c^2} \nu^2 \Re(\mathcal{P}_x) \\
& - \frac{k\nu}{2\epsilon_0 c^2} \Re(\mathcal{M}_y)
\end{aligned} \tag{25}$$

As $\mathcal{E}, \mathcal{B}, \mathcal{M}, \mathcal{P}$, ϕ_E and ϕ_B do not change appreciably in an optical frequency period, we use the slowly varying amplitude and phase approximation and obtain the equations below. Also, substituting these values in Eq. (25) and separating the real and imaginary parts, we obtain the equations below.

$$k \frac{\partial \mathcal{E}_x}{\partial z} + \frac{\nu}{c^2} \frac{\partial \mathcal{E}_x}{\partial t} = -\kappa \mathcal{E} - \frac{1}{2\epsilon_0 c^2} \nu^2 \Im(\mathcal{P}_x) - \frac{k\nu}{2\epsilon_0 c^2} \Im(\mathcal{M}_y) \tag{26}$$

$$k \frac{\partial \phi_{Ex}}{\partial z} + \frac{\nu}{c^2} \frac{\partial \phi_{Ex}}{\partial t} = k^2 - \frac{\nu^2}{c^2} - \frac{1}{2\epsilon_0 c^2 \mathcal{E}_x} \nu^2 \Re(\mathcal{P}_x) - \frac{k\nu}{2\epsilon_0 c^2 \mathcal{E}_x} \Re(\mathcal{M}_y). \tag{27}$$

A similar derivation gives us the equations for the magnetic field:

$$k \frac{\partial \mathcal{B}_y}{\partial z} + \frac{\nu}{c^2} \frac{\partial \mathcal{B}_y}{\partial t} = -\kappa \mathcal{B}_y + \frac{1}{2\epsilon_0 c^2} k^2 \Im(\mathcal{M}_y) + \frac{k\nu}{2\epsilon_0 c^2} \Im(\mathcal{P}_x). \tag{28}$$

$$k \frac{\partial \phi_{By}}{\partial z} + \frac{\nu}{c^2} \frac{\partial \phi_{By}}{\partial t} = k^2 - \frac{\nu^2}{c^2} + \frac{1}{2\epsilon_0 c^2 \mathcal{B}_y} k^2 \Re(\mathcal{M}_y) + \frac{k\nu}{2\epsilon_0 c^2 \mathcal{B}_y} \Re(\mathcal{P}_x). \tag{29}$$

Here $\kappa = \sigma/2\epsilon_0 c$ is the linear loss coefficient.

These equations can be compared with the equations that are generally used in the study of the interactions of atoms with matter.

$$\frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t} = -\kappa \mathcal{E} - \frac{1}{2\epsilon_0} k \Im(\mathcal{P}) \tag{30}$$

$$\frac{\partial \phi_E}{\partial z} + \frac{1}{c} \frac{\partial \phi_E}{\partial t} = k - \frac{\nu}{c} - \frac{1}{2\epsilon_0} k \mathcal{E}^{-1} \Re(\mathcal{P}) \tag{31}$$

and

$$\frac{\partial \mathcal{B}}{\partial z} + \frac{1}{c} \frac{\partial \mathcal{B}}{\partial t} = -\kappa \mathcal{B} + \frac{1}{2\epsilon_0 c^2} k \Im(\mathcal{M}) \tag{32}$$

$$\frac{\partial \phi_B}{\partial z} + \frac{1}{c} \frac{\partial \phi_B}{\partial t} = k - \frac{\nu}{c} + \frac{1}{2\epsilon_0 c^2} k \mathcal{B}^{-1} \Re(\mathcal{M}) \tag{33}$$

-
- [1] J. Kastel, M.Fleischauer, S. Yelin, and R. Walsworth, Phys. Rev. Lett. **99**, 073602 (2007).
 - [2] V. Veselago, Sov. Phys. Usp **10**, 509 (1968).
 - [3] J. Pendry, A. Holden, D. Robbins, and W. Stewart, IEEE Trans. Micro. Theory Tech. **47**, 2075 (1999).
 - [4] R. Shelby, D. R. Smith, and S. Schultz, Science **292**, 77 (2001).
 - [5] V. Shalaev, Nature Photonics **1**, 41 (2007).
 - [6] C. M. Soukoulis, S. Linden, and M. Wegener, Science **315**, 47 (2007).
 - [7] T. J. Yen, W. J. Padilla, N. Fang, D. C. Vier, D. R. Smith, J. B. Pendry, D. N. Basov, and X. Zhang, Science **303**, 1494 (2004).
 - [8] S. Linden, C. Enkrich, M. Wegener, J. Zhou, T. Koschny, and C. M. Soukoulis, Science **303**, 1351 (2004).
 - [9] P. V. Parimi, W. T. Lu, P. Vodo, J. Sokoloff, J. S. Derov, and S. Sridhar, Phys. Rev. Lett. **92**, 127401 (2004).
 - [10] A. Berrier, M. Mulot, M. Swillo, M. Qiu, L. Thylen, A. Talne, and A. Anand, Phys. Rev. Lett. **93**, 073902 (2004).
 - [11] J. Pendry, Science **306**, 1353 (2004).
 - [12] Z. J. Simmons, N. A. Proite, J. Miles, D. E. Sikes, and D. D. Yavuz, Phys. Rev. B **85**, 0583810 (2012).
 - [13] J. Kastel, M.Fleischauer, S. Yelin, and R. Walsworth, Phys. Rev. B **79**, 063818 (2009).
 - [14] T. O'Dell, *The Electrodynamics of Magneto-Electric Media* (Elsevier, NY, 1970).
 - [15] H. J. McGuinness, M. G. Raymer, C. J. McKinstrie, and S. Radic, PRL **105**, 093604 (2010).
 - [16] M. Johnsson and M. Fleischauer, Phys. Rev. B **66**, 043808 (2002).
 - [17] M. Johnsson and M. Fleischauer, Phys. Rev. B **68**,

- 023804 (2003).
- [18] A. K. Popov, S. A. Myslivets, T. F. George, and V. M. Shalaev, Optics Letters **32**, 3044 (2008).
- [19] M. D. Lukin, A. B. Matsko, M. Fleischhauer, and M. O. Scully, PRL **82**, 1847 (1999).
- [20] A change in principal quantum number n leads to much more forbidden magnetic dipole transitions